1. How many multiples of 20 are also divisors of 17!?

   Answer: 7056

   Solution: $17! = 2^{15} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13 \times 17$. A multiple of 20 needs to be a multiple of $2^2 \times 5$. Thus, the number of possible combinations of the factors are $14 \times 7 \times 3 \times 3 \times 2 \times 2 \times 2 = \boxed{7056}$

2. Suppose for some positive integers, that $\frac{p+\frac{1}{q}}{q+\frac{1}{p}} = 17$. What is the greatest integer n such that $\frac{p+q}{n}$ is always an integer?

   Answer: 18

   Solution: If we multiply both the top and bottom of the fraction, we'll get $\frac{p^2q+p}{q^2p+q}$, which can be factored as $\frac{p(pq+1)}{q(pq+1)} = \frac{p}{q}$, which means that we now have $\frac{p}{q} = 17$. Since $p = 17q$, $p + q = 18q$, so 18 will always divide $p + q$.

3. Find the minimal N such that any N-element subset of $\{1, 2, 3, 4, \ldots 7\}$ has a subset S such that the sum of elements of S is divisible by 7.

   Answer: 4

   Solution: We see that $N > 3$ since $1+2+3 = 6 < 7$. We see that $N \leq 4$ since any four element subset of $\{1, 2, 3, \ldots, 7\}$ has elements x, $-x$.

4. What is the remainder when 201820182018... [2018 times] is divided by 15?

   Answer: 13

   Solution: The sum of the digits is 11*2018 $\equiv$ 2 * 2 (mod 3) $\equiv$ 1 (mod 3). Since the last digit is 8, the number is $\equiv 3 \pmod 5$. By the Chinese Remainder Theorem, the number is $\equiv 13 \pmod{15}$.

5. If r_i are integers such that $0 \leq r_i < 31$ and r_i satisfies the polynomial $x^4 + x^3 + x^2 + x \equiv 30 \pmod{31}$, find

   $$\sum_{i=1}^{4} (r_i^2 + 1)^{-1} \pmod{31}$$

   where x^{-1} is the modulo inverse of x, that is, it is the unique integer y such that $0 < y < 31$ and $xy - 1$ is divisible by 31.

   Answer: 2

   Solution: We can easily list out the squares modulo 31 to compute that $\sqrt{5} = 6$. Thus, dividing both sides by x^2 and moving the constant term to the other side, we have

   $$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} \equiv \left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1 \equiv 0 \pmod{31}$$

   Letting $u = x + \frac{1}{x}$ and solving the quadratic, we have

   $$u \equiv 18, 12 \pmod{31}$$

   Note that if r is a root, then either

   $$r + \frac{1}{r} \equiv \frac{r^2+1}{r} \equiv 18 \pmod{31} \implies \frac{1}{r^2+1} = \frac{1}{18r}$$

or

$$r + \frac{1}{r} \equiv \frac{r^2+1}{r} \equiv 12 \pmod{31} \implies \frac{1}{r^2+1} = \frac{1}{12r}$$

Summing up, we have

$$1 + 1 \equiv 2 \equiv \boxed{29} \pmod{31}$$

6. Ankit wants to create a pseudo-random number generator using modular arithmetic. To do so he starts with a seed x_0 and a function $f(x) = 2x + 25 \pmod{31}$. To compute the kth pseudo random number, he calls $g(k)$ defined as follows:

$$g(k) = \begin{cases} x_0 & \text{if } k = 0 \\ f(g(k-1)) & \text{if } k > 0 \end{cases}$$

If x_0 is 2017, compute $\sum_{j=0}^{2017} g(j) \pmod{31}$.

Answer: 21

Solution: We show that the function g is periodic with period 5. Given $f(x) = ax+b \pmod{m}$, if $a^y \equiv 1 \pmod{m}$ we will show that as long as $a \neq 1$, $g(k)$ relative to f is periodic with period y. $g(y) = a^y \cdot x + b\sum_{i=0}^{i=y-1} a^i$. We will show that the summation in this term is 0 as long as $a \neq 1$.

Proof. Let $S \equiv 1 + a + a^2 + \cdots + a^{x-1} \pmod{m}$. Then $aS \equiv a + a^2 + a^3 + \cdots + a^x \pmod{m}$. However, since $a^x = 1$, $aS \equiv 1 + a + a^2 + \cdots + a^{x-1} \equiv S \pmod{m}$. This means that $aS \equiv S \pmod{m}$. Since $a \neq 1$, $S = 0$. □

So we have $g(y) = a^y \cdot x_0 = x_0$. Therefore, $g(k+y) = g(y)$ for all k. We now only need to compute the value x such that $2^x \equiv 1 \pmod{31}$. This is clearly 5. Plugging in for values $i = 1...i = 5$. we get $2, 29, 25, 5, 4$ in modulo 31, which is $61 \equiv -1 \pmod{31}$. Summing up to 2014 yields $(3 \cdot 403)$. Adding them to the values of g from $2015, 2016, 2017$ we get our final answer of $\boxed{25}$

7. Determine the number of ordered triples (a, b, c), with $0 \leq a, b, c \leq 10$ for which there exists (x, y) such that $ax^2 + by^2 \equiv c \pmod{11}$

Answer: 1221

Solution: Note that if a, b, c are not all divisible by 11, then there exists a solution since the set

$$\{ax^2 | x \in \mathbb{Z}_{11}\}$$

has 6 elements and the set

$$\{c - by^2 | y \in \mathbb{Z}_{11}\}$$

also has 6 elements. Therefore, the two sets have a nonempty intersection. We now count the number of non-triples, that is triples for which there do not exist such x and y. First, if $a = b = 0$, all values of $c \neq 0$ is a nonsolution, giving a total of 10 in this case. Now suppose $a = 0$ and $b \neq 0$. We will multiply by 2 to account for symmetry. The tuple is a nonsolution if and only if cb^{-1} is not a quadratic residue. There are a total of 5 quadratic residues, so for a given $c \neq 0$, there exists 5 values of b that give a nonsolution. Thus, in this case, there are

$5 \times 10 = 50$ nonsolutions. Multiplying by 2 gives 100 nonsolutions. Finally, if $c = 0$, then there are no non-solutions since one can choose $x = y = 0$. This gives 110 total non-solutions. There are $11^3 = 1331$ pairs, and subtracting the nonsolutions, we arrive at $1331 - 110 = \boxed{1221}$

8. How many $1 < n \leq 2018$ such that the set $\{0, 1, 1+2, \ldots, 1+2+3+\cdots+i, \ldots, 1+2+\cdots+n-1\}$ is a permutation of $\{0, 1, 2, 3, 4, \cdots, n-1\}$ when reduced modulo n?

Answer: 10

Solution: We first claim that all $n = 2^k$ work. To show this, suppose not. Then there exists a sequence $a + a + 1 + a + 2 + \cdots + a + l = (l + 1) + \frac{l(l+1)}{2} \equiv 0 \pmod{2^k}$. If l is odd, then $l + 1 + \frac{l(l+1)}{2} = \frac{l+1}{2}(2 + l) \equiv 0 \pmod{2^k}$. But l is odd, so $2 + l$ is odd, so $\frac{l+1}{2} \equiv 0 \pmod{2^k}$. This is a contradiction to $l < 2^k$. If l is even, we have $(l+1)\left(1 + \frac{l}{2}\right) \equiv 0 \pmod{2^k}$, so $1 + \frac{l}{2} \equiv 0 \pmod{2^k}$, again contradicting $l < 2^k$. Hence, $n = 2^k$ works. Now, if n has an odd divisor greater than 1, let $2m + 1$ be the minimal odd divisor of n (e.g. a prime). Let $n = (2m + 1)k$. Then

$$\sum_{i=k-m}^{k+m} i = \frac{(2m + 1)2k}{2} = (2m + 1)k = n$$

Since there are 11 powers of two less than or equal to 2018, the answer is $\boxed{10}$.

9. Compute the following:

$$\sum_{x=0}^{99} (x^2 + 1)^{-1} \pmod{199}$$

where x^{-1} is the value $0 \leq y \leq 199$ such that $xy - 1$ is divisible by 199.

Answer: 150

Solution: Note that 199 is prime.

Step 1: polynomial division

First, let us perform long division $\frac{x^{198}-1}{x^2+1}$

$$x^{198} - 1 = P(x)(x^2 + 1) + c$$

First, we see that the remainder must be of even degree, since if not, then P must be an odd function, but we see that $P(x)$ has a x^{196} term, a contradiction. Hence, the remainder is an even function, so the remainder is of an even degree, and therefore it is a constant. Secondly, we have $x^2 \equiv -1$, so $x^{198} - 1 = (x^2)^{99} - 1 = -1 - 1 = -2$. Therefore,

$$x^{198} - 1 = P(x)(x^2 + 1) - 2$$

Step 2: Destroying the quotient

Let p be a prime. The following is a well-known fact:

$$\sum_{n=1}^{p-1} n^k \equiv 0 \pmod{p}$$

if $p \neq 1 \pmod{k}$

Proof. Let g be a generator of $\mathbb{Z}_p$. Then

$$\sum_{n=1}^{p-1} n^k \equiv \sum_{n=0}^{p-2} g^{kn} \equiv \frac{g^{k(p-1)}-1}{g^k-1} \equiv 0 \pmod p$$

as desired.

Step 3: Putting everything together
Note that since $199 \equiv 3 \pmod 4$ that the denominator of $\frac{1}{x^2+1}$ is never 0 as $x \in \mathbb{Z}_{199}$. Hence,

$$-1 \equiv \sum_{x=0}^{198} \frac{x^{198}-1}{x^2+1} \equiv \sum_{x=0}^{198} P(x) - \sum_{x=0}^{198} \frac{2}{x^2+1} \equiv 0 - \sum_{x=0}^{198} \frac{2}{x^2+1} \pmod{199}$$

This implies that

$$\sum_{x=0}^{198} \frac{2}{x^2+1} \equiv 1 \pmod{199}$$

or that

$$\sum_{x=0}^{99} \frac{1}{x^2+1} \equiv -\frac{1}{4} \equiv \boxed{150} \pmod{199}$$

□

10. Evaluate the following

$$\prod_{j=1}^{50} \left(2\cos\left(\frac{4\pi j}{101}\right)+1\right)$$

Answer: −1

Solution: Let $q = \exp\left(\frac{2\pi}{101}\right)$, and let $n = \frac{2\pi}{101}$. Then

$$2\cos\left(\frac{4\pi j}{101}\right)+1 = q^{2j} + q^{-2j} + 1 = \frac{q^{3j}-q^{-3j}}{q^j-q^{-j}}$$

Hence, the product is equal to

$$\prod_{j=1}^{50} \frac{q^{3j}-q^{-3j}}{q^j-q^{-j}}$$

Note that as 3 "acts" on the set $S = \{1,2,3,4,\ldots,50\}$ by multiplication, some elements get sent to $a \in S$ or $-a$ where $a \in S$. Note that if $3i \equiv 3j \pmod{101} \implies i \equiv j$, and furthermore that if $3i \equiv -3j \pmod{101} \implies i \equiv -j$, so i, j are not both in S. Hence, $3 \cdot S$ is equal to S except some elements are negative of what they were. Note that if $j \equiv -i \pmod{101}$, then $q^{3j} - q^{-3j} = -(q^{3i} - q^{-3i})$. Hence, the product is equal to the product of all the negative signs in $3S$. We just need to determine if there are an even number or if there are an odd number of negative signs in $3S$. Note that $3\cdot1, 3\cdot2, 3\cdot3, \ldots, 3\cdot16$ are all positive (all in S), $3\cdot17, 3\cdot19, 3\cdot20, \ldots 3\cdot21, \ldots, 3\cdot33$ are all negative (in the complement of S), $3\cdot34, 3\cdot37, \ldots, 3\cdot50$ are all positive. Therefore, there are $33-17+1 = 17$ negative signs, and thus the product is $\boxed{-1}$