1. Find the maximum integral value of k such that $0 \leq k \leq 2019$ and

$$|e^{2\pi i \frac{k}{2019}} - 1|$$

is maximal.

Answer: 1010

Solution: Geometrically, this is the length of the secant from 1 to $e^{2\pi i \frac{k}{n}}$. The maximal integral values occur when $k = \frac{2019-1}{2} = 1009$ or $k = \frac{2019+1}{2} = \boxed{1010}$, the latter of which is our answer.

2. Find the remainder when 2^{2019} is divided by 7.

Answer: 1

Solution: By Fermat's Little Theorem, $2^{7-1} = 2^6 \equiv 1 \pmod 7$. Since $2^{2019} = 2^{6\cdot 336+3}$, $2^{2019} \equiv 1 \cdot 2^3 = 8 \equiv \boxed{1} \pmod 7$.

3. A cylinder with radius 5 and height 1 is rolling on the (unslanted) floor. Inside the cylinder, there is water that has constant height $\frac{15}{2}$ as the cylinder rolls on the floor. What is the volume of the water?

Answer: $\mathbf{\frac{25\sqrt{3}}{4} + \frac{50\pi}{3}}$ **or** $\mathbf{\frac{75\sqrt{3}+200\pi}{12}}$

Solution: We have $10 - \frac{15}{2} = \frac{20-15}{2} = \frac{5}{2}$, so the angle the water does not cover fully is equal to $\frac{\pi}{3}$ radians. We then realize that the $\frac{\pi}{3}$ portion that the water does cover is a triangle with side lengths 5, 5, and $5\sqrt{3}$. That triangle has area equal to $\frac{25\sqrt{3}}{4}$, so the volume of the cylinder (since the height is 1) is $\boxed{\frac{25\sqrt{3}}{4} + \frac{50\pi}{3}}$ as desired.

4. Let C be the number of ways to arrange the letters of the word CATALYSIS, T be the number of ways to arrange the letters of the word TRANSPORT, S be the number of ways to arrange the letters of the word STRUCTURE, and M be the number of ways to arrange the letters of the word MOTION. What is $\frac{C-T+S}{M}$?

Answer: 126

Solution: We find that $C = \frac{9!}{2!2!}$, $T = \frac{9!}{2!2!}$, $S = \frac{9!}{2!2!2!}$, and $M = \frac{6!}{2!}$. Since $C = T$, $\frac{C-T+S}{M} = \frac{S}{M} = \frac{\frac{9!}{2!2!2!}}{\frac{6!}{2!}} = \frac{9\cdot 8\cdot 7}{2\cdot 2} = \boxed{126}$.

5. What is the minimum distance between $(2019, 470)$ and $(21a - 19b, 19b + 21a)$ for $a, b \in \mathbb{Z}$?

Answer: $\mathbf{\sqrt{101}}$

Solution: The method is as follows: we first solve the equation and we find integers as close to it as possible. We have $21a - 19b = 2019$, $19b + 21a = 470$. This gives us $42a = 2489$ and $38b = -1549$. This gives us $a \sim 59.2$, $b \sim -40.7$, so picking $a = 59$ and $b = -41$, we find that $21a - 19b = 1239 + 779 = 2018$ and $19b + 21a = 1239 - 779 = 460$, so the minimum distance is $\sqrt{1^2 + 10^2} = \boxed{\sqrt{101}}$.

6. At a party, 2019 people decide to form teams of three. To do so, each turn, every person not on a team points to one other person at random. If three people point to each other (that is, A points to B, B points to C, and C points to A), then they form a team. What is the probability that after 65,536 turns, exactly one person is not on a team?

   Answer: 0

   Solution: Since 2019 is divisible by 3, there is no way that exactly one person is not on a team of three. Therefore, there is exactly $\boxed{0}$ chance that this ends up happening.

7. How many distinct ordered pairs of integers (b, m, t) satisfy the equation $b^8+m^4+t^2+1 = 2019$?

   Answer: 16

   Solution: First, we subtract 1 from both sides to get the equation $b^8 + m^4 + t^2 = 2018$. Since $3^8 = 6561 > 2018$ and all even powers of integers are positive, we know that $|b| = 0$, 1, or 2. Even powers must also be congruent to 0 or 1 (mod 4), and $2018 \equiv 2 \pmod 4$, so two of b, m, and t are odd. If $b = 0$, we must find odd m and t such that $m^4 + t^2 = 2018$. Since $7^4 = 2401 > 2018$, $|m|$ must equal 1, 3, or 5. Clearly, $|m| \neq 1$ because 2017 is not a square, and $2018 - 3^4 = 1937$ and $2018 - 5^4 = 1393$ are not squares, so $b \neq 0$. If $b = \pm 1$, we need to find integers t and m of opposing parities such that $m^4 + t^2 = 2018 - 1 = 2017$. Plugging in values of m from 0 to 6, we find that $2017 - 3^4 = 1936 = 44^2$, yielding us 8 ordered pairs (accounting for signs). If $b = \pm 2$, then m and t are odd integers that satisfy $m^4 + t^2 = 2018 - 2^8 = 1762$. Plugging in $m = 1$, 3, and 5, we find that $1762 - 3^4 = 1681 = 41^2$, yielding another 8 ordered pairs. Thus, there are $8 + 8 = \boxed{16}$ distinct ordered pairs of integers (b, m, t) that satisfy the equation $b^8 + m^4 + t^2 + 1 = 2019$.

8. Let (k_i) be a sequence of unique nonzero integers such that $x^2 - 5x + k_i$ has rational solutions. Find the minimum possible value of
$$\frac{1}{5}\sum_{i=1}^{\infty}\frac{1}{k_i}.$$

   Answer: $\mathbf{-\frac{137}{1500}}$ **or** $\mathbf{-0.091\overline{3}}$

   Solution: If α and β are roots of this polynomial, then we have $\alpha + \beta = 5$ and $\alpha\beta = k_i$. Then the sum we want to find is
$$\frac{1}{2}\cdot\frac{1}{5}\left(\sum_{x=6}^{\infty}\frac{1}{x(5-x)} + \sum_{x=-\infty}^{-1}\frac{1}{x(5-x)}\right) = \boxed{-\frac{137}{1500}}$$
   since the sum telescopes and since each root occurs twice.

9. You wish to color every vertex, edge, face, and the interior of a cube one color each such that no two adjacent objects are the same color. Faces are adjacent if they share an edge. Edges are adjacent if they share a vertex. The interior is adjacent to all of its faces, edges, and vertices. Each face is adjacent to all of its edges and vertices. Each edge is adjacent to both of its vertices. What is the minimum number of colors required to do this?

   Answer: 5

   Solution: Let's represent each vertex in the cube with a bit string from 000 to 111 (This can actually be done for any cube of arbitrary dimension; you can think of it as the vertex labeled 010 is at point $(0, 1, 0)$ in 3D space). Furthermore, we notice that any edge, face, or the interior can be interpreted as a set of vertices sharing a certain number of coordinates. Let's denote by x a free coordinate. Therefore, $0x1$ represents the edge connecting $(0,0,1)$ and $(0,1,1)$, and $x0x$ represents the face composed of vertices 000, 001, 100, 101. (This means that there are 3^n total things we need to consider, where n is the dimension of the cube)

Under this notation, we can reduce the adjacency rules to the following: Two sets are adjacent if:

(a) They have the same number of xs, all but one of their numbers are the same, and one x is in a different spot. Therefore, $1x0$ is adjacent to $x10$ and $11x$, but not $1x1$ or $x01$.

(b) They have different numbers of xs, and the spots with numbers are the same. Therefore, $1xx$ is adjacent to 101, $10x$ and xxx, but not $0x1$.

I can find a set of 5 strings which are all adjacent to each other (the interior, three faces, and their shared vertex; in the former form, xxx, $xx1$, $x1x$, $1xx$, 111). Since those must all be different colors, we need at least $\boxed{5}$ colors.

The following shows that 5 colors are sufficient. We need one color for the interior, and since it touches everything, nothing else can be that color. Furthermore, we need at least three distinct colors for edges, and three colors for faces. Note, however, that a face and an edge can share a vertex and not be adjacent. Therefore, we can make a "jail cell" the same color, consisting of two opposite faces and their connecting edges. We can thus color all edges and faces with 3 colors, and use a fifth color for the vertices.

The coloring described above is:

- Red: xxx
- Blue: $xx1, xx0, 11x, 10x, 01x, 00x$
- Green: $x1x, x0x, 1x1, 1x0, 0x1, 0x0$
- Cyan: $1xx, 0xx, x11, x10, x01, x00$
- Yellow: $111, 110, 101, 100, 011, 010, 001, 000$

As an aside, this problem seems to become rather difficult with higher-dimension cubes. I don't know of any solutions that generalizes perfectly to higher dimensions; I've at least found an upper bound of colors required of $2^{n-1}+1$, but only a lower bound around $n+2$. I'd be interested in hearing a general solution to this!

10. Compute the remainder when the product of all positive integers less than and relatively prime to 2019 is divided by 2019.

Answer: 1

Solution: Let the product be P. We will compute $P \mod 3, 673$ and use the Chinese Remainder Theorem to compute $P \mod 2019$. Since $673 \equiv 1 \pmod 3$ and $1346 \equiv 2 \pmod 3$, so

$$P = (1 \cdot 2 \cdot 4 \cdot 5 \cdots 670 \cdot 671) \cdot (674 \cdot 676 \cdot 677 \cdots 1343 \cdot 1345) \cdots (1348 \cdot 1349 \cdots 2017 \cdot 2018)$$

Take each product $\mod 673$ and we get

$$P = (1 \cdot 2 \cdot 4 \cdot 5 \cdots 670 \cdot 671) \cdot (1 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdots 670 \cdot 672) \cdot (2 \cdot 3 \cdot 5 \cdot 6 \cdots 671 \cdot 672)$$

Regrouping gives us $P = (672!)^2 \equiv (-1)^2 \pmod{673}$ by Wilson's Theorem since 673 is a prime. We know that there is 673 natural numbers less than 2019 that is $\equiv 1 \pmod 3$ and 672 numbers that is $\equiv 2 \pmod 3$ We then compute

$$P \equiv 1^{673} \cdot 2^{672} \equiv \left(2^2\right)^{336} \equiv 1^{336} \equiv 1 \pmod 3$$

Since $P \equiv 1 \pmod 3$ and $P \equiv 1 \pmod{673}$, applying the Chinese Remainder Theorem gives us $P \equiv 1 \pmod{2019}$.

11. A baseball league has 64 people, each with a different 6-digit binary number whose base-10 value ranges from 0 to 63. When any player bats, they do the following: for each pitch, they swing if their corresponding bit number is a 1; otherwise, they decide to wait and let the ball pass. For example, the player with the number 11 has binary number 001011. For the first and second pitch, they wait; for the third, they swing, and so on. Pitchers follow a similar rule to decide whether to throw a splitter or a fastball; if the bit is 0, they will throw a splitter, and if the bit is 1, they will throw a fastball.

    If a batter swings at a fastball, then they will score a hit; if they swing on a splitter, they will miss and get a "strike." If a batter waits on a fastball, then they will also get a strike. If a batter waits on a splitter, then they get a "ball." If a batter gets 3 strikes, then they are out; if a batter gets 4 balls, then they automatically get a hit. For example, if player 11 pitched against player 6 (binary is 000110), the batter would get a ball for the first pitch, a ball for the second pitch, a strike for the third pitch, a strike for the fourth pitch, and a hit for the fifth pitch; as a result, they will count that as a "hit." If player 11 pitched against player 5 (binary is 000101), however, then the fifth pitch would be the batter's third strike, so the batter would be "out."

    Each player in the league plays against every other player exactly twice; once as batter, and once as pitcher. They are then given a score equal to the number of outs they throw as a pitcher plus the number of hits they get as a batter. What is the highest score received?

    Answer: 63

    Solution: Interestingly enough, for every pair of people, if you win as a pitcher, then you end up losing as a batter. This is because the calls for a pitching against b is exactly the same as the calls for b pitching against a: a strike is called if one of the binary digits is a 0 and the other is a 1, a ball if both are 0, and a hit of both are 1. As a result, every person ends up scoring exactly $\boxed{63}$ points.

12. 2019 people (all of whom are perfect logicians), labeled from 1 to 2019, partake in a paintball duel. First, they decide to stand in a circle, in order, so that Person 1 has Person 2 to his left and person 2019 to his right. Then, starting with Person 1 and moving to the left, every person who has not been eliminated takes a turn shooting. On their turn, each person can choose to either shoot one non-eliminated person of his or her choice (which eliminates that person from the game), or deliberately miss. The last person standing wins. If, at any point, play goes around the circle once with no one getting eliminated (that is, if all the people playing decide to miss), then automatic paint sprayers will turn on, and end the game with everyone losing. Each person will, on his or her turn, always pick a move that leads to a win if possible, and, if there is still a choice in what move to make, will prefer shooting over missing, and shooting a person closer to his or her left over shooting someone farther from their left. What is the number of the person who wins this game? Put "0" if no one wins.

    Answer: 1991

Solution: Note that no one will shoot someone that is not directly to his or her left, the last person will always prefer shooting to making everyone lose, and shooting someone when there are n people remaining essentially means that you are putting yourself as the last person in an $n-1$-person game.

Let's look at the case for 3 people, A, B, and C. If A decides to miss, then B will also decide to miss, C will shoot A, and B would win. On the other hand, if A decides to shoot, then he is immediately shot by C, and so loses. Since either way, he loses, A decides to shoot B, thus leading person C to win.

For 4 people, person 1 decides to shoot first, as that guarantees that he wins, because then he would be the equivalent of person C in a three person game. Therefore, person 1 wins a 4 person game. Note that if person n wins an n person game, then person 1 will win a $n+1$ person game for this reason.

For 5 people, assume persons 1 through 4 decide to miss. Then 5 will shoot 1, and thus person 2 would win. Since that means that 4 will lose whether or not they shoot, 4 will decide to shoot 5 instead of miss. That would cause 1 to win, so 3 would lose whether or not he shoots. Going backwards, that means that person 1 will decide to shoot, since missing would cause him to lose anyway. As a result, person 3 would win. In general, if person $k \neq n$ is going to win an n person game, then person $k+2$ will win an $n+1$ person game.

One can therefore show that person 5 wins in 6 people, and 7 wins with 7 people. A pattern seems to be emerging; indeed, person 1 wins every game where $n = 2^k$ for some integer k. That means that person 1 wins with 2048 people. Therefore, person 2047 wins with 2047 people, and $2047 - 28 \cdot 2 = \boxed{1991}$ wins with $2047 - 28 = 2019$ people.

13. Triangle $\triangle ABC$ has $AB = 13$, $BC = 14$, and $CA = 15$. $\triangle ABC$ has incircle γ and circumcircle ω. γ has center at I. Line $\overline{AI}$ is extended to hit ω at P. What is the area of quadrilateral $ABPC$?

Answer: 112

Solution: Let E and F be on $\overline{BC}$ such that $\overline{AE} \perp \overline{BC}$ and $\overline{PF} \perp \overline{BC}$, and let $\overline{AP} \cap \overline{BC} = D$. We can compute via Heron's formula that the area of $\triangle ABC$ is 84, $AE = 12$, and $BE = 5$. In addition, via the angle bisector theorem, we can compute that $BD = 6.5$ and $DE = 1.5$. Finally, we realize that F is the midpoint of $\overline{BC}$, so $AF = 7$ and $DF = 0.5$. Since $\triangle AED \sim \triangle PFD$, we realize that $\frac{AE}{PF} = 3$. Thus, the area of $\triangle BPC$ is $\frac{4 \cdot 14}{2} = 28$, and the area of $ABPC$ is $84 + 28 = \boxed{112}$.

14. A regular hexagon has positive integer side length. A laser is emitted from one of the hexagon's corners, and is reflected off the edges of the hexagon until it hits another corner. Let a be the distance that the laser travels. What is the smallest possible value of a^2 such that $a > 2019$? You need not simplify/compute exponents.

Answer: $\mathbf{2019^2 + 3}$ or 4076364

Solution: Note that when a line reflects off a mirror, the result is identical if we imagine that the room it was in was reflected instead, and the line kept moving straight. Since a hexagon has symmetry and tesselates the plane, we can imagine a hexagonal grid, and a line travelling over the grid until it hits a corner. By the question, we want that line to be barely longer than 2019 units. All corners of hexagons rest on a triangular grid, so let's see if we can determine any restrictions on the distance of a triangle vertex from the origin. Let $a = (0, 1)$ and $b = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Then any triangle vertex is a linear combination of a and b. Some algebra gives you that the distance of this point is $\sqrt{a^2 + ab + b^2}$, where a and b are integers. Therefore, any path will take $\sqrt{n}$ distance, for integer n. $\sqrt{2019^2 + 1}$ and $\sqrt{2019^2 + 2}$ are both impossible because a^2+ab+b^2 can't be 2 (mod 4) nor 2 (mod 3). However, $\sqrt{2019^2 + 3}$ is possible, with $a = 2018, b = 2$. We still need to check that this yields a valid hexagon corner. As it turns out, when the hexagon has side length 2, that point yields a valid corner, and the path the laser takes does not hit any previous corner. Therefore, our answer is $2019^2 + 3 = \boxed{4076364}$.

15. A group of aliens from Gliese 667 Cc come to Earth to test the hypothesis that mathematics is indeed a universal language. To do this, they give you the following information about their mathematical system:

- For the purposes of this experiment, the Gliesians have decided to write their equations in the same syntactic format as in Western math. For example, in Western math, the expression "5+4" is interpreted as running the "+" operation on numbers 5 and 4. Similarly, in Gliesian math, the expression $\alpha\gamma\beta$ is interpreted as running the "γ" operation on numbers α and β.
- You know that γ and η are the symbols for addition and multiplication (which works the same in Gliesian math as in Western math), but you don't know which is which. By some bizarre coincidence, the symbol for equality is the same in Gliesian math as it is in Western math; equality is denoted with an "=" symbol between the two equal values.
- Two symbols that look exactly the same have the same meaning. Two symbols that are different have different meanings and, therefore, are not equal.

They then provide you with the following equations, written in Gliesian, which are known to be true:

$$\begin{array}{ccc} \pitchfork\eta\rhd = \curlywedge & \circledcirc\gamma\varkappa = \gtrdot & \ltimes\gamma\gtrdot = \curlywedge \\ \gtrdot\gamma\Diamond = \rhd & \gtrdot\eta\Cup = \Diamond & \gtrdot\eta\ltimes = \varkappa \\ \square\gamma\gtrdot = \pitchfork & \pitchfork\eta\Cup = \square & \square\eta\Cup = \gtrdot \end{array}$$

What is the human number equivalent of $\circledcirc$?

Answer: $\mathbf{\frac{1}{3}}$ **or** $\mathbf{0.\overline{3}}$

Solution: A natural first step would be to try and determine whether $\gamma = +$ or $\eta = +$. In terms of properties, the one thing that distinguishes addition from multiplication is the distributive property; that is, $(a + b) \cdot c = ac + bc$, but $(a \cdot b) + c$ may not equal $(a + c) \cdot (b + c)$. Using the distributive property would require a set of three equations using the same operation, and sharing at least one character. Two characters fit that requirement; $\gtrdot$ with γ, and $\Cup$ with η.

Let's focus on $\Cup$. We see that the three values η'd into $\Cup$ are $\gtrdot, \pitchfork$, and $\square$. Since those would correspond to the a, b, and $a + b$ in the distributive property, we should search for an equation with $\square$, $\gtrdot$, $\pitchfork$, and γ. There is one of those, fortunately; that one says that $\square\gamma\gtrdot = \pitchfork$. If we assume η is multiplication and γ is addition, then that yields the statement $(\square + \gtrdot) \cdot \Cup = \pitchfork \cdot \Cup = \square$, and $\square \cdot \Cup + \gtrdot \cdot \Cup = \gtrdot + \Diamond = \rhd$. This would imply that $\gtrdot = \Diamond$. Therefore, η cannot equal multiplication, so $\eta = +$ and $\gamma = \cdot$.

If we were to focus on $\gtrdot$, then we could try something similar; the three values γ'd into $\gtrdot$ are $\Diamond$, $\Box$, and $\ltimes$. Unfortunately, no equation uses all three of those symbols, so we can't get any information from that process.

Now that we know that $\eta = +$ and $\gamma = \cdot$, let's try to solve this problem. Since we are looking for $\circledcirc$, we should focus on the equation using $\circledcirc$; the only one using that is $\circledcirc \cdot \varkappa = \gtrdot$. Let's try substituting this as far as we can, until only $\circledcirc$ and $\gtrdot$ remain. Note that $\gtrdot$ can't equal zero, since it is multiplied into numbers to yield results not equal to $\gtrdot$:

$$\begin{aligned}
\circledcirc \cdot \varkappa &= \gtrdot \\
\circledcirc \cdot (\gtrdot + \ltimes) &= \gtrdot \\
\circledcirc \cdot \frac{\gtrdot + \curlywedge}{\gtrdot} &= \gtrdot, \text{ since } \gtrdot \text{ can't be } 0 \\
\circledcirc \cdot \left(\gtrdot + \frac{\pitchfork + \rhd}{\gtrdot}\right) &= \gtrdot \\
\circledcirc \cdot \left(\gtrdot + \frac{\Box \cdot \gtrdot + \gtrdot \cdot \Diamond}{\gtrdot}\right) &= \gtrdot \\
\circledcirc \cdot (\gtrdot + (\Box + \Diamond)) &= \gtrdot \\
\circledcirc \cdot (\gtrdot + (\Box + \gtrdot + \Cup)) &= \gtrdot \text{ Note: This is the only step where multiple equations can be used.} \\
\circledcirc \cdot (\gtrdot + (\gtrdot + \gtrdot)) &= \gtrdot \\
\circledcirc \cdot (3 \cdot \gtrdot) &= \gtrdot \\
\circledcirc &= \boxed{\frac{1}{3}} \text{ as desired.}
\end{aligned}$$