1. Let x be a real number such that $x^2 - x + 1 = 7$ and $x^2 + x + 1 = 13$. Compute the value of x^4.

   Answer: 81

   Solution: Subtracting the first equation from the second yields $2x = 6$, which implies $x = 3$. Thus, $x^4 = 3^4 = \boxed{81}$.

2. Let f and g be linear functions such that $f(g(2021)) - g(f(2021)) = 20$. Compute $f(g(2022)) - g(f(2022))$. (Note: A function h is linear if $h(x) = ax + b$ for all real numbers x.)

   Answer: 20

   Solution: For real numbers a, b, c, and d, let $f(x) = ax + b$, and let $g(x) = cx + d$. Observe that

   $$f(g(x)) - g(f(x)) = ad + b - bc - d,$$

   so this value is constant for each x. Therefore, the answer is $\boxed{20}$.

3. Let x be a solution to the equation $\lfloor x \lfloor x + 2 \rfloor + 2 \rfloor = 10$. Compute the smallest C such that for any solution x, $x < C$. Here, $\lfloor m \rfloor$ is defined as the greatest integer less than or equal to m. For example, $\lfloor 3 \rfloor = 3$ and $\lfloor -4.25 \rfloor = -5$.

   Answer: $\mathbf{\frac{9}{4}}$

   Solution: If $\lfloor x \lfloor x + 2 \rfloor + 2 \rfloor = 10$, then $x \lfloor x + 2 \rfloor + 2 < 11$, which means that $x \lfloor x + 2 \rfloor < 9$. To do some bounding, recognize that if $x = 2$, then $x \lfloor x + 2 \rfloor = 8$. In addition, if $x = 3$, then $x \lfloor x + 2 \rfloor = 15$. Thus, for our inequality to be adhered, we must have $2 < x < 3$, which means that $\lfloor x \rfloor = 2$. Thus, our expression becomes $4x < 9 \implies x < \frac{9}{4}$, so the smallest possible value of C is $\boxed{\dfrac{9}{4}}$.

4. Let θ be a real number such that $1 + \sin 2\theta - \left(\frac{1}{2} \sin 2\theta\right)^2 = 0$. Compute the maximum value of $(1 + \sin\theta)(1 + \cos\theta)$.

   Answer: 1

   Solution: Let $S = \sin\theta + \cos\theta$ and $P = \sin\theta\cos\theta$. We can see that the value which we wish to compute is $1 + S + P$. By sine properties, we see that

   $$1 + \sin 2\theta - \left(\frac{1}{2}\sin 2\theta\right)^2 = 1 + 2\sin\theta\cos\theta - (\sin\theta\cos\theta)^2 = 1 + 2P - P^2 = 0,$$

   so $P = 1 \pm \sqrt{2}$. However, P can't be greater than 1, since sine and cosine have an upper bound of 1, so $P = 1 - \sqrt{2}$. Expanding the original equation slightly differently yields

   $$\begin{aligned} 1 + 2\sin\theta\cos\theta - (\sin\theta\cos\theta)^2 &= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - (\sin\theta\cos\theta)^2 \\ &= (\sin\theta + \cos\theta)^2 - (\sin\theta\cos\theta)^2 \\ &= S^2 - P^2. \end{aligned}$$

   As a result, we see that $1 + 2P - P^2 = S^2 - P^2$, so $S = \pm(1 - \sqrt{2})$. We want to maximize $1 + S + P$, and since P is fixed, this is equivalent to maximizing S. Thus, we get $S = \sqrt{2} - 1$, and hence, $1 + S + P = 1 + (\sqrt{2} - 1) + (1 - \sqrt{2}) = \boxed{1}$, which is our answer.

5. Compute the sum of the real solutions to $\lfloor x \rfloor \{x\} = 2020x$. Here, $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x, and $\{x\} = x - \lfloor x \rfloor$.

Answer: $-\frac{1}{2021}$

Solution: Noting that $x = \{x\} + \lfloor x \rfloor$, we can simplify the equation into $\lfloor x \rfloor \{x\} = 2020\lfloor x \rfloor + 2020\{x\}$. By Simon's Favorite Factoring Trick, this factors to

$$(\lfloor x \rfloor - 2020)(\{x\} - 2020) = 2020^2.$$

However, we note that, because $0 \le \{x\} < 1$, we have $-2020 \le \{x\} - 2020 < -2019$. Then $-\dfrac{2020^2}{2019} < \lfloor x \rfloor - 2020 \le -2020$, so $-\dfrac{2020}{2019} < \lfloor x \rfloor \le 0$. However, $\lfloor x \rfloor$ is an integer, so it must be either 0 or -1. If $\lfloor x \rfloor = 0$, then we find that $x = 0$ is a solution. If $\lfloor x \rfloor = -1$, then we can substitute this into the original expression to get $-1(x+1) = 2020x$ where solving yields $x = -\dfrac{1}{2021}$. Thus, the sum of the solutions is $-\dfrac{1}{2021} + 0 = \boxed{-\dfrac{1}{2021}}$.

6. Let f be a real function such that for all $x \neq 0, x \neq 1$,

$$f(x) + f\left(-\frac{1}{x-1}\right) = \frac{9}{4x^2} + f\left(1 - \frac{1}{x}\right).$$

Compute $f\left(\frac{1}{2}\right)$.

Answer: $\frac{45}{8}$

Solution: The main motivation behind the problem is that $g(x) = 1 - \frac{1}{x}$ cycles as $x \to 1 - \frac{1}{x} \to \frac{-1}{x-1} \to x$. Given this, recognize that plugging in x and $1 - \frac{1}{x}$ gives us the following equations side by side:

$$f(x) + f\left(-\frac{1}{x-1}\right) - f\left(1 - \frac{1}{x}\right) = \frac{9}{4x^2} = \left(\frac{3}{2x}\right)^2$$

$$f\left(1 - \frac{1}{x}\right) + f(x) - f\left(-\frac{1}{x-1}\right) = \frac{9}{4\left(1 - \frac{1}{x}\right)^2} = \left(\frac{3x}{2(x-1)}\right)^2.$$

Adding the equations together gives $2f(x) = \left(\frac{3}{2x}\right)^2 + \left(\frac{3x}{2(x-1)}\right)^2$ and dividing by 2 yields $f(x) = \frac{1}{2}\left(\left(\frac{3}{2x}\right)^2 + \left(\frac{3x}{2(x-1)}\right)^2\right)$. Evaluating this at $x = \frac{1}{2}$, we get $f\left(\frac{1}{2}\right) = \boxed{\frac{45}{8}}$.

7. Let $z_1, z_2, \ldots, z_{2020}$ be the roots of the polynomial $z^{2020} + z^{2019} + \cdots + z + 1$. Compute

$$\sum_{i=1}^{2020} \frac{1}{1 - z_i^{2020}}.$$

Answer: **1010**

Solution: First note that if z is a root of the given polynomial, then z is a root of

$$(z-1)(z^{2020} + z^{2019} + \cdots + z + 1) = z^{2021} - 1.$$

Hence, the values z_i are the 2021st roots of unity except for 1. Because 2020 is relatively prime to 2021, the values z_i^{2020} are simply a permutation of the values z_i. That is to say,

$$\sum_{i=1}^{2020} \frac{1}{1-z_i^{2020}} = \sum_{i=1}^{2020} \frac{1}{1-z_i}.$$

Now observe that the solution set $\{z_i\}$ is contained entirely in the circle $|z| = 1$ and is symmetric about the real axis. This means that the set $\{1 - z_i\}$ is contained entirely in the circle $|z| = 2\cos(\arg z)$ (i.e. the polar graph $r = 2\cos\theta$) and is also symmetric about the real axis. Thus, the set $\left\{\frac{1}{1-z_i}\right\}$ is contained entirely in the set $|z| = \frac{1}{2}\sec(\arg z)$, and again, it is symmetric about the real axis. The set $|z| = \frac{1}{2}\sec(\arg z)$ is better identified as the set $\mathrm{Re}(z) = \frac{1}{2}$ (by multiplying each side of the equation by $\cos(\arg z)$), which means that

$$\mathrm{Re}\sum_{i=1}^{2020} \frac{1}{1-z_i} = 2020 \cdot \frac{1}{2} = 1010.$$

Further, since this set is symmetric about the real axis, the imaginary part of the sum is equal to 0, so the answer is $\boxed{1010}$.

8. Let $f(w) = w^3 - rw^2 + sw - \frac{4\sqrt{2}}{27}$ denote a polynomial, where $r^2 = \left(\frac{8\sqrt{2}+10}{7}\right)s$. The roots of f correspond to the sides of a right triangle. Compute the smallest possible area of this triangle.

Answer: $\frac{\sqrt[3]{2}}{9}$

Solution: The roots must be in the form a, b, and $\sqrt{a^2+b^2}$. Then $a + b + \sqrt{a^2+b^2} = r$ and $ab + a\sqrt{a^2+b^2} + b\sqrt{a^2+b^2} = ab + (a+b)\sqrt{a^2+b^2} = s$. Let $x = \sqrt{a^2+b^2}$, $y = a+b$, $z = ab$. Thus, $x+y = r$ and $xy+z = s$. Note that $x^2+2z = y^2$, so $z = \frac{y^2-x^2}{2}$, and thus $y^2+2xy-x^2 = 2s$. Now, let $r^2 = \alpha s$, so that $\alpha = \frac{8\sqrt{2}+10}{7}$. Then $(x+y)^2 = \frac{\alpha}{2}(y^2+2xy-x^2)$, or

$$\left(1+\frac{\alpha}{2}\right)x^2 + (2-\alpha)xy + \left(1-\frac{\alpha}{2}\right)y^2 = 0.$$

Note that solutions must be in the form of $x = ky$, as any solution (x, y) will have a corresponding solution (mx, my), where m is some real number. Hence, plugging $x = ky$ and dividing by y, we get

$$\left(1+\frac{\alpha}{2}\right)k^2 + (2-\alpha)k + \left(1-\frac{\alpha}{2}\right) = 0.$$

Now, plugging back in α, we get the equation $\left(\frac{12+4\sqrt{2}}{7}\right)k^2 + \left(\frac{4-8\sqrt{2}}{7}\right)k + \left(\frac{2-4\sqrt{2}}{7}\right) = 0$ or, simplifying out the 7's in the denominator, $(12+4\sqrt{2})k^2 + (4-8\sqrt{2})k + (2-4\sqrt{2}) = 0$. Solving the quadratic for k, we obtain

$$\begin{aligned}
k &= \frac{8\sqrt{2}-4 \pm \sqrt{\left(8\sqrt{2}-4\right)^2 + 4\left(12+4\sqrt{2}\right)\left(4\sqrt{2}-2\right)}}{2\left(12+4\sqrt{2}\right)} \\
&= \frac{8\sqrt{2}-4 \pm \sqrt{176+96\sqrt{2}}}{2\left(12+4\sqrt{2}\right)} \\
&= \frac{8\sqrt{2}-4 \pm (12+4\sqrt{2})}{2\left(12+4\sqrt{2}\right)}.
\end{aligned}$$

Note that since both x and y are positive, k must also be positive, so we take the positive solution. Hence, $k = \frac{8\sqrt{2}-4+(12+4\sqrt{2})}{2(12+4\sqrt{2})} = \frac{12\sqrt{2}+8}{2(12+4\sqrt{2})} = \frac{1}{\sqrt{2}}$. Hence we have $x = \frac{y}{\sqrt{2}}$. Now, by AM-GM,

$$\frac{1}{\sqrt{2}}x = \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} = \frac{y}{2}.$$

Thus, $x \geq \frac{y}{\sqrt{2}}$ with equality at $a = b$. Using Vieta's on the condition given by the coefficient of the product term, we get $ab\sqrt{a^2+b^2} = a^3\sqrt{2} = \frac{4\sqrt{2}}{27} \implies a = \frac{\sqrt[3]{4}}{3}$. Thus, we have $a = b = \frac{\sqrt[3]{4}}{3}$, so the area is $\frac{1}{2}ab = \boxed{\frac{\sqrt[3]{2}}{9}}$.

9. Compute the sum of the positive integers $n \leq 100$ for which the polynomial $x^n + x + 1$ can be written as the product of at least 2 polynomials of positive degree with integer coefficients.

Answer: 1648

Solution: If a polynomial $p(x)$ is reducible, then it may be written in the form $p(x) = f(x)g(x)$. The polynomial $p'(x) = x^{\deg(p)}p(1/x)$ is p but with the coefficients reversed. Suppose p and p' do not share any roots. Then $p'(x) = f'(x)g'(x)$, so $pp' = kk'$, where $k = \pm fg'$ and $k \neq \pm p, p'$. For p any polynomial, notice that the coefficient of x^n of pp' is the sum of the squares of the coefficients of p. Substituting $p(x) = x^n + x + 1$, we find that the coefficient of x^n in pp' is 3, indicating that k must be a sum of 3 monomials:

$$(x^n + x^{n-1} + 1)(x^n + x + 1) = x^{2n} + x^{2n-1} + x^{n+1} + 3x^n + x^{n-1} + x + 1.$$

Since the top coefficient of pp' is x^{2n} and the bottom coefficient is 1, k must be of the form $(-1)^{p_1}x^n + (-1)^{p_2}x^a + (-1)^{p_1}$. Multiplying kk' out, we get

$$\begin{aligned} kk' &= ((-1)^{p_1}x^n + (-1)^{p_2}x^a + (-1)^{p_1})\left((-1)^{p_1}x^n + (-1)^{p_2}x^{n-a} + (-1)^{p_1}\right) \\ &= x^{2n} + (-1)^{p_1+p_2}x^{2n-a} + 3x^n + (-1)^{p_1+p_2}x^{n+a} + (-1)^{p_1+p_2}x^a + (-1)^{p_1+p_2}x^{n-a} + 1 \end{aligned}$$

so $x^{2n-1} + x^{n+1} + x^{n-1} + x = (-1)^{p_1+p_2}(x^{n+a} + x^a + x^{n-a} + x^{2n-a})$. It becomes clear that a must equal 1 or $n-1$ and $p_2 = p_1$.

Thus, if $x^n + x + 1$ and $x^n + x^{n-1} + 1$ share no roots, then they are irreducible. Conversely, if they do share roots, then these polynomials will have a nontrivial common factor if $n > 2$ and hence not be irreducible. Therefore, we notice that any roots of those two polynomials must be a root of $x^{n-2} - 1$. Let ω be such a root. Then $\omega^n + \omega + 1 = \omega^2 + \omega + 1 = 0$, so ω must be a third root of unity and so $n - 2 \equiv 0 \pmod{3}$. Thus, $x^n + x + 1$ is irreducible if and only if $n = 2$ or $n \not\equiv 2 \pmod{3}$. Summing all desired $n \leq 100$ up, we get $16(5 + 98) = \boxed{1648}$.

10. Given a positive integer n, define $f_n(x)$ to be the number of square-free positive integers k such that $kx \leq n$. Then, define $v(n)$ as

$$v(n) = \sum_{i=1}^{n}\sum_{j=1}^{n} f_n\left(i^2\right) - 6f_n\left(ij\right) + f_n\left(j^2\right).$$

Compute the largest positive integer $2 \leq n \leq 100$ for which $v(n) - v(n-1)$ is negative. (Note: A square-free positive integer is a positive integer that is not divisible by the square of any prime.)

Answer: 60

Solution: For some positive integer n, denote $p(n)$ to be the number of distinct prime factors of n. Note that $p(1) = 0$. Furthermore, denote $\mathcal{F}(x)$ to be the number of square-free positive integers less than or equal to x for any nonnegative real number x. We first prove two lemmas.

Lemma 1. *For any nonnegative real number x and positive integer $a \geq \lfloor x \rfloor$, $\displaystyle\sum_{i=1}^{a} \mathcal{F}\left(\frac{x}{i^2}\right) = \lfloor x \rfloor$.*

Proof. Note that because a square-free number k counted in $\mathcal{F}\left(\frac{x}{i^2}\right)$ must satisfy $ki^2 \leq x$, the sum $\displaystyle\sum_{i=1}^{\lfloor x \rfloor} \mathcal{F}\left(\frac{x}{i^2}\right)$ counts the number of ways a positive integer less than or equal to x can be represented as the product of a square-free number and a square. Since for any such n, there exists exactly one way to represent $n = ki^2$, where k is square-free and i is a positive integer, we achieve $\displaystyle\sum_{i=1}^{\lfloor x \rfloor} \mathcal{F}\left(\frac{x}{i^2}\right) = \lfloor x \rfloor$.

Then, note that for any $i > \lfloor x \rfloor$ we have $i^2 \geq i > x$, meaning that $\frac{x}{i^2} < 1$. Since the smallest square-free number is 1, we have $\mathcal{F}\left(\frac{x}{i^2}\right) = 0$, meaning $\displaystyle\sum_{i=1}^{a} \mathcal{F}\left(\frac{x}{i^2}\right) = \sum_{i=1}^{\lfloor x \rfloor} \mathcal{F}\left(\frac{x}{i^2}\right) + \sum_{i=\lfloor x \rfloor+1}^{a} \mathcal{F}\left(\frac{x}{i^2}\right) = \sum_{i=1}^{\lfloor x \rfloor} \mathcal{F}\left(\frac{x}{i^2}\right) = \lfloor x \rfloor$, as desired. □

Lemma 2. *For any nonnegative real number x and positive integer $a \geq \lfloor x \rfloor$, $\displaystyle\sum_{i=1}^{a} \mathcal{F}\left(\frac{x}{i}\right) = \sum_{i=1}^{\lfloor x \rfloor} 2^{p(i)}$.*

Proof. Similar to the proof of **Lemma 1**, we can note that a square-free number k counted in $\mathcal{F}\left(\frac{x}{i}\right)$ must satisfy $ki \leq x$, and therefore, the sum $\displaystyle\sum_{i=1}^{\lfloor x \rfloor} \mathcal{F}\left(\frac{x}{i}\right)$ counts the number of ways a positive integer less than or equal to x can be represented as the product of a square-free number and a positive integer. For each positive integer n, consider $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ to be the prime factorization of n, where $p_i \neq p_j$ for $i \neq j$, $e_i > 0$ for all i, and $k = p(n)$. Then, note that the number of ways n can be represented as the product of a square-free number and a positive integer is simply equal to the number of square-free factors of n. Because a square-free factor of n is of the form $n' = p_1^{e'_1} p_2^{e'_2} \cdots p_k^{e'_k}$, where $e'_i \leq 1$ for all i, there are exactly $2^k = 2^{p(n)}$ square-free factors of n. Summing over all $n \leq x$, we obtain that $\displaystyle\sum_{i=1}^{\lfloor x \rfloor} \mathcal{F}\left(\frac{x}{i}\right) = \sum_{i=1}^{\lfloor x \rfloor} 2^{p(i)}$.

Then, note that for any $i > \lfloor x \rfloor$, we have $i > x$, meaning $\frac{x}{i} < 1$. Since the smallest square-free number is 1, we have $\mathcal{F}\left(\frac{x}{i}\right) = 0$, meaning $\displaystyle\sum_{i=1}^{a} = \sum_{i=1}^{\lfloor x \rfloor} \mathcal{F}\left(\frac{x}{i}\right) + \sum_{i=\lfloor x \rfloor+1}^{a} \mathcal{F}\left(\frac{x}{i}\right) = \sum_{i=1}^{\lfloor x \rfloor} \mathcal{F}\left(\frac{x}{i}\right) = \sum_{i=1}^{\lfloor x \rfloor} 2^{p(i)}$, as desired. □

Now, note that a square-free positive integer k is only counted by $f_n(x)$ if and only if $k \leq \frac{n}{x}$; therefore, $f_n(x) = \mathcal{F}\left(\frac{n}{x}\right)$. Thus, we can rewrite $v(n)$ as

$$\begin{aligned} v(n) &= \sum_{i=1}^{n}\sum_{j=1}^{n} f_n(i^2) - 6f_n(ij) + f_n(j^2) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{F}\left(\frac{n}{i^2}\right) - 6\mathcal{F}\left(\frac{n}{ij}\right) + \mathcal{F}\left(\frac{n}{j^2}\right) \\ &= 2n\sum_{i=1}^{n} \mathcal{F}\left(\frac{n}{i^2}\right) - 6\sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{F}\left(\frac{n}{ij}\right) \\ &= 2n\lfloor n\rfloor - 6\sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{F}\left(\frac{\left(\frac{n}{i}\right)}{j}\right) \\ &= 2n^2 - 6\sum_{i=1}^{n}\sum_{j=1}^{\lfloor \frac{n}{i}\rfloor} 2^{p(j)} \\ &= 2n^2 - 6\sum_{i=1}^{n}\left\lfloor\frac{n}{i}\right\rfloor 2^{p(i)} \\ &= 2n^2 - 6\sum_{i=1}^{n}\sum_{d\mid i} 2^{p(d)} \\ &= 6\left(\frac{n^2}{3} - \sum_{i=1}^{n}\sum_{d\mid i} 2^{p(d)}\right) \end{aligned}$$

by applying our two lemmas. Now, we can simply subtract $v(n) - v(n-1)$ to obtain

$$\begin{aligned} v(n) - v(n-1) &= 6\left(\frac{n^2}{3} - \sum_{i=1}^{n}\sum_{d\mid i} 2^{p(d)}\right) - 6\left(\frac{(n-1)^2}{3} - \sum_{i=1}^{n-1}\sum_{d\mid i} 2^{p(d)}\right) \\ &= 6\left(\frac{2n-1}{3} - \sum_{d\mid n} 2^{p(d)}\right) \end{aligned}$$

It then suffices to find the maximum $n \geq 2$ such that $\sum_{d\mid n} 2^{p(d)} > \frac{2n-1}{3}$. For any positive integer n, let its prime factorization be $p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$, where $p_i \neq p_j$ for $i \neq j$, $e_i > 0$ for all i, and $k = p(n)$. We will then denote $E(n) = \prod_{i=1}^{k}(2e_i + 1)$ and claim that $\sum_{d\mid n} 2^{p(d)} = E(n)$ for all n.

Proof. Consider the generating function $\prod_{i=1}^{k}\left(1 + 2p_i + 2p_i^2 + \cdots + 2p_i^{e_i}\right)$. Each factor

$$n' = p_1^{e'_1}p_2^{e'_2}\cdots p_k^{e'_k}$$

of n is represented by exactly one term in the expansion. Furthermore, note that for every prime factor p_i, the coefficient of n' is multiplied by 2 if p_i divides n' and is multiplied by 1 otherwise. Thus, the coefficient of n' in the expansion of $\prod_{i=1}^{k}\left(1+2p_i+2p_i^2+\cdots+2p_i^{e_i}\right)$ is exactly $2^{p(n')}$. Then, to compute the value of $\sum_{d|n} 2^{p(d)}$, we want to find the sum of all coefficients of the generating function, which is simply

$$\prod_{i=1}^{k}\left(1+\underbrace{2+2+\cdots+2}_{e_i \text{ times}}\right)=\prod_{i=1}^{k}(2e_i+1)=E(n),$$

as desired. □

Now, we want to find the maximum n such that $E(n) > \frac{2n-1}{3}$. Heuristically, $E(n)$ is maximal when n contains many prime factors. Simply by testing different distributions of prime factors, we can see that the maximal possible value of n is 60.

Prime distribution	$E(n)$	$\max(n)$
$\{6\}$	$13 \leq \frac{2\cdot 64-1}{3}$	Not possible
$\{5,1\}$	$33 \leq \frac{2\cdot 96-1}{3}$	Not possible
$\{5\}$	$11 \leq \frac{2\cdot 32-1}{3}$	Not possible
$\{4,1\}$	$27 \leq \frac{2\cdot 48-1}{3}$	Not possible
$\{3,2\}$	$35 \leq \frac{2\cdot 72-1}{3}$	Not possible
$\{4\}$	$9 \leq \frac{2\cdot 16-1}{3}$	Not possible
$\{3,1\}$	21	24
$\{2,1,1\}$	45	60

Since $\frac{2\cdot 60-1}{3}=\frac{119}{3}$, we do not need to test any value n for which $E(n)\leq 39$, meaning $\boxed{60}$ is the maximum possible value of n, and we are done.

Remark. It is provable that 60 is the maximum value of n in general without imposing an upper bound of 100. The inequality $\prod_{i=1}^{k}(2e_i+1) > p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ limits n due to size reasons, as the left-hand side is linear in e_i, while the right-hand side is exponential.