Time limit: 90 minutes.
Instructions: This test contains 25 short answer questions.
No calculators.

1. Carson and Emily attend different schools. Emily's school has four times as many students as Carson's school. The total number of students in both schools combined is 10105. How many students go to Carson's school?

2. Let x be a real number such that $x^2 - x + 1 = 7$ and $x^2 + x + 1 = 13$. Compute the value of x^4.

3. A scalene acute triangle has angles whose measures (in degrees) are whole numbers. What is the smallest possible measure of one of the angles, in degrees?

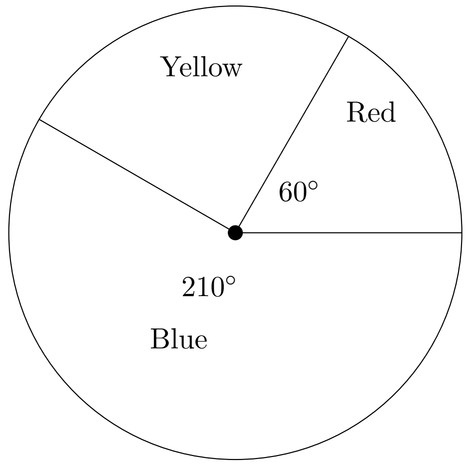
4. Moor and Samantha are drinking tea at a constant rate. If Moor starts drinking tea at 8:00am, he will finish drinking 7 cups of tea by 12:00pm. If Samantha joins Moor at 10:00am, they will finish drinking the 7 cups of tea by 11:15am. How many hours would it take Samantha to drink 1 cup of tea?

5. Bill divides a 28×30 rectangular board into two smaller rectangular boards with a single straight cut, so that the side lengths of both boards are positive whole numbers. How many different pairs of rectangular boards, up to congruence and arrangement, can Bill possibly obtain? (For instance, a cut that is 1 unit away from either of the edges with length 28 will result in the same pair of boards: either way, one would end up with a 1×28 board and a 29×28 board.)

6. A toilet paper roll is a cylinder of radius 8 and height 6 with a hole in the shape of a cylinder of radius 2 and the same height. That is, the bases of the roll are annuli with inner radius 2 and outer radius 8. Compute the surface area of the roll.

7. Alice is counting up by fives, starting with the number 3. Meanwhile, Bob is counting down by fours, starting with the number 2021. How many numbers between 3 and 2021, inclusive, are counted by both Alice and Bob?

8. On the first day of school, Ashley the teacher asked some of her students what their favorite color was and used those results to construct the pie chart pictured below. During this first day, 165 students chose yellow as their favorite color. The next day, she polled 30 additional students and was shocked when none of them chose yellow. After making a new pie chart based on the combined results of both days, Ashley noticed that the angle measure of the sector representing the students whose favorite color was yellow had decreased. Compute the difference, in degrees, between the old and the new angle measures.



9. Rakesh is flipping a fair coin repeatedly. If T denotes the event where the coin lands on tails and H denotes the event where the coin lands on heads, what is the probability Rakesh flips the sequence HHH before the sequence THH?

10. Triangle $\triangle ABC$ has side lengths $AB = AC = 27$ and $BC = 18$. Point D is on $\overline{AB}$ and point E is on $\overline{AC}$ such that $\angle BCD = \angle CBE = \angle BAC$. Compute DE.

11. Compute the number of sequences of five positive integers $a_1, \ldots, a_5$ where all $a_i \leq 5$ and the greatest common divisor of all five integers is 1.

12. Let a, b, and c be the solutions of the equation
$$x^3 - 3 \cdot 2021^2 x = 2 \cdot 2021^3.$$
Compute $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

13. A six-sided die is rolled four times. What is the probability that the minimum value of the four rolls is 4?

14. Let $r_1, r_2, \ldots, r_{47}$ be the roots of $x^{47} - 1 = 0$. Compute
$$\sum_{i=1}^{47} r_i^{2020}.$$

15. Benji has a 2×2 grid, which he proceeds to place chips on. One by one, he places a chip on one of the unit squares of the grid at random. However, if at any point there is more than one chip on the same square, Benji moves two chips on that square to the two adjacent squares, which he calls a chip-fire. He keeps adding chips until there is an infinite loop of chip-fires. What is the expected number of chips that will be added to the board?

16. Jason and Valerie agree to meet for game night, which runs from 4:00 PM to 5:00 PM. Jason and Valerie each choose a random time from 4:00 PM to 5:00 PM to show up. If Jason arrives first, he will wait 20 minutes for Valerie before leaving. If Valerie arrives first, she will wait 10 minutes for Jason before leaving. What is the probability that Jason and Valerie successfully meet each other for game night?

17. Simplify $\sqrt[4]{17+12\sqrt{2}} - \sqrt[4]{17-12\sqrt{2}}$.

18. In quadrilateral $ABCD$, suppose that $\overline{CD}$ is perpendicular to $\overline{BC}$ and $\overline{DA}$. Point E is chosen on segment $\overline{CD}$ such that $\angle AED = \angle BEC$. If $AB = 6$, $AD = 7$, and $\angle ABC = 120^\circ$, compute $AE + EB$.

19. How many three-digit numbers $\underline{abc}$ have the property that when it is added to $\underline{cba}$, the number obtained by reversing its digits, the result is a palindrome? (Note that $\underline{cba}$ is not necessarily a three-digit number since before reversing, c may be equal to 0.)

20. For some positive integer n, $(1+i)+(1+i)^2+(1+i)^3+\cdots+(1+i)^n = (n^2-1)(1-i)$, where $i=\sqrt{-1}$. Compute the value of n.

21. There exist integers a and b such that $\left(1+\sqrt{2}\right)^{12} = a + b\sqrt{2}$. Compute the remainder when ab is divided by 13.

22. Austin is at the Lincoln Airport. He wants to take 5 successive flights whose destinations are randomly chosen among Indianapolis, Jackson, Kansas City, Lincoln, and Milwaukee. The origin and destination of each flight may not be the same city, but Austin must arrive back at Lincoln on the last of his flights. Compute the probability that the cities Austin arrives at are all distinct.

23. Shivani has a single square with vertices labeled $ABCD$. She is able to perform the following transformations:

   - She does nothing to the square.
   - She rotates the square by 90, 180, or 270 degrees.
   - She reflects the square over one of its four lines of symmetry.

   For the first three timesteps, Shivani only performs reflections or does nothing. Then for the next three timesteps, she only performs rotations or does nothing. She ends up back in the square's original configuration. Compute the number of distinct ways she could have achieved this.

24. Given that x, y, and z are a combination of positive integers such that $xyz = 2(x+y+z)$, compute the sum of all possible values of $x+y+z$.

25. Let $\triangle BMT$ be a triangle with $BT = 1$ and height 1. Let O_0 be the centroid of $\triangle BMT$, and let $\overline{BO_0}$ and $\overline{TO_0}$ intersect $\overline{MT}$ and $\overline{BM}$ at B_1 and T_1, respectively. Similarly, let O_1 be the centroid of $\triangle B_1MT_1$, and in the same way, denote the centroid of $\triangle B_nMT_n$ by O_n, the intersection of $\overline{BO_n}$ with $\overline{MT}$ by B_{n+1}, and the intersection of $\overline{TO_n}$ with $\overline{BM}$ by T_{n+1}. Compute the area of quadrilateral $MBO_{2021}T$.