1. Compute
$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^2(x)+\frac{\mathrm{d}}{\mathrm{d}x}\cos^2(x).$$

Answer: 0

Solution: We compute
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sin^2(x)+\frac{\mathrm{d}}{\mathrm{d}x}\cos^2(x)&=\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^2(x)+\cos^2(x)\right)\\&=\frac{\mathrm{d}}{\mathrm{d}x}(1)\\&=\boxed{0}.\end{aligned}$$

2. Let $f(x)=(x-2)(x-7)^2+2x$. Compute the unique real number c not equal to 7 such that $f'(c)=f'(7)$.

Answer: $\frac{11}{3}$

Solution: First of all, by the product rule,
$$\begin{aligned}f'(x)&=\left((x-7)^2+(x-2)\cdot 2(x-7)\right)+2\\&=(3x-11)(x-7)+2.\end{aligned}$$

One strategy would be to expand out. Alternatively, note that a real number c has $f'(c)=f'(7)$ if and only if
$$\begin{aligned}(3c-11)(c-7)&=f'(c)-2\\&=f'(7)-2\\&=(3\cdot 7-11)(7-7)\\&=0.\end{aligned}$$

This gives $c=7$ or $c=11/3$, so the answer is $\boxed{\frac{11}{3}}$.

3. Compute
$$\int_0^1 e^{x+e^x+e^{e^x}}\,\mathrm{d}x.$$

Answer: $e^{e^e}-e^e$

Solution: Chain Rule tells us
$$\frac{\mathrm{d}}{\mathrm{d}x}e^{e^{e^x}}=e^xe^{e^x}e^{e^{e^x}}.$$

Hence,
$$\begin{aligned}\int_0^1 e^xe^{e^x}e^{e^{e^x}}\,\mathrm{d}x&=\left[e^{e^{e^x}}\right]_0^1\\&=\boxed{e^{e^e}-e^e}\end{aligned}$$

as desired.

4. Let $f(x)$ be a degree-4 polynomial such that $f(x)$ and $f'(x)$ both have 20 and 22 as roots. Given that $f(21) = 21$, compute $f(23)$.

   Answer: 189

   Solution: We claim that $f(x)$ has double roots at both 20 and 22. Indeed, suppose a real number a has $f(a) = f'(a) = 0$. Because $f(a) = 0$, we may factor $f(x) = (x-a)g(x)$ for some polynomial $g(x)$. Applying the product rule, we see

$$f'(x) = (x-a)g'(x) + g(x),$$

   so $f'(a) = 0$ forces $g(a) = 0$. In particular, we may factor $g(x) = (x-a)h(x)$ for some polynomial $h(x)$, so $f(x) = (x-a)^2h(x)$.

   Applying the above argument with $a = 20$ and $a = 22$, we may write $f(x) = (x-20)^2(x-22)^2r(x)$ for some polynomial $r(x)$. However, $f(x)$ has degree 4, so $r(x) = c$ for some real number c. Using the fact that $f(21) = 21$, we see

$$\begin{aligned} c &= c(21-20)^2(21-22)^2 \\ &= f(21) \\ &= 21. \end{aligned}$$

   Thus, $f(x) = 21(x-20)^2(x-22)^2$, so $f(23) = 21 \cdot 9 \cdot 1 = \boxed{189}$.

5. Compute

$$\sum_{n=0}^{\infty} \left(\sqrt{n^2+3n+2} - \sqrt{n^2+n} - 1 \right).$$

   Answer: $\frac{1}{2}$

   Solution: Let S be the value of the series, and let $S_k = \sum_{n=0}^{k} \left[\sqrt{n^2+3n+2} - \sqrt{n^2+n} - 1 \right]$. Note that $\lim_{k\to\infty} S_k = S$. For a given k, we compute

$$\begin{aligned} S_k &= \sum_{n=0}^{k} \left[\sqrt{(n+1)(n+2))} - \sqrt{n(n+1)} - 1 \right] \\ &= \sum_{n=0}^{k} \sqrt{(n+1)(n+2)} - \sum_{n=0}^{k} \sqrt{n(n+1)} + \sum_{n=0}^{k} (-1) \\ &= \sum_{n=1}^{k+1} \sqrt{n(n+1)} - \sum_{n=0}^{k} \sqrt{n(n+1)} + \sum_{n=0}^{k} (-1) \\ &= \sqrt{(k+1)(k+2)} - (k+1) \end{aligned}$$

   Taking the limit,

$$\begin{aligned} S &= \lim_{k\to\infty} S_k \\ &= \lim_{k\to\infty} \sqrt{(k+1)(k+2)} - (k+1) \\ &= \lim_{k\to\infty} \frac{k+1}{\sqrt{(k+1)(k+2)} + k + 1} \\ &= \boxed{\frac{1}{2}}. \end{aligned}$$

6. Compute

$$\int_0^{\pi/3} \sec(x)\sqrt{\tan(x)\sqrt{\tan(x)\sqrt{\tan(x)\sin(x)}}}\,\mathrm{d}x.$$

Answer: $\mathbf{\frac{8}{7}\left(2^{7/8}-1\right)}$

Solution: We simplify the inside of the radical. To begin, let I be the value of the integral. We note

$$\begin{aligned} I &= \int_0^{\pi/3} \frac{\tan(x)}{\sin(x)}\sqrt{\tan(x)\sqrt{\tan(x)\sqrt{\tan(x)\sin(x)}}}\,\mathrm{d}x \\ &= \int_0^{\pi/3} \tan(x)\sqrt{\sec(x)\sqrt{\sec(x)\sqrt{\sec(x)}}}\,\mathrm{d}x \\ &= \int_0^{\pi/3} \frac{\tan(x)\sec(x)\sqrt{\sec(x)\sqrt{\sec(x)\sqrt{\sec(x)}}}}{\sec(x)}\,\mathrm{d}x\,. \end{aligned}$$

Setting $u = \sec(x)$, we see $\mathrm{d}u = \tan(x)\sec(x)\,\mathrm{d}x$, so the integral becomes

$$\int_1^2 \frac{\sqrt{u\sqrt{u\sqrt{u}}}}{u}\,\mathrm{d}u = \int_1^2 u^{-1/8}\,\mathrm{d}u.$$

Applying the power rule, we see

$$\begin{aligned} \int_1^2 u^{-1/8}\,\mathrm{d}u &= \left[\frac{u^{7/8}}{7/8}\right]_1^2 \\ &= \frac{2^{7/8}-1}{7/8} \\ &= \boxed{\frac{8}{7}\left(2^{7/8}-1\right)}. \end{aligned}$$

7. Compute

$$\lim_{x\to 0}\left(1+\int_0^x \frac{\cos(t)-1}{t^2}\,\mathrm{d}t\right)^{1/x}.$$

Answer: $\mathbf{\frac{1}{\sqrt{e}}}$

Solution: To begin, we set

$$f(x) = \int_0^x \frac{\cos(t)-1}{t^2}\,\mathrm{d}t\,.$$

In particular, $f(0) = 0$ and $f'(x) = (\cos(x)-1)/x^2$. In fact, applying L'Hôpital's rule, we compute

$$\begin{aligned} \lim_{x\to 0} f'(x) &= \lim_{x\to 0} \frac{-\sin(x)}{2x} \\ &= \lim_{x\to 0} \frac{-\cos(x)}{2} \\ &= -\frac{1}{2}. \end{aligned}$$

We now proceed with the solution. The key idea is to write

$$\begin{aligned}\lim_{x\to 0}\left(1+\int_0^x \frac{\cos(t)-1}{t^2}\,\mathrm{d}t\right)^{1/x} &= \lim_{x\to 0}(1+f(x))^{1/x}\\ &= \exp\left(\lim_{x\to 0}\ln\left((1+f(x))^{1/x}\right)\right)\\ &= \exp\left(\lim_{x\to 0}\frac{\ln(1+f(x))}{x}\right),\end{aligned}$$

where ln denotes the natural logarithm and exp denotes the exponential function. Applying L'Hôpital's rule to the limit, we see

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1+f(x))}{x} &= \lim_{x\to 0}\frac{f'(x)/(1+f(x))}{1}\\ &= \lim_{x\to 0}\frac{f'(x)}{1+f(x)}\\ &= \frac{-1/2}{1+0}\\ &= -\frac{1}{2}.\end{aligned}$$

Thus, our answer is $\exp\left(-\frac{1}{2}\right) = \boxed{\frac{1}{\sqrt{e}}}$.

8. At the Berkeley Mart for Technology, every item has a real-number cost independently and uniformly distributed from 0 to 2022. Sumith buys different items at the store until the total amount he spends strictly exceeds 1. Compute the expected value of the number of items Sumith buys.

   Answer: $\boldsymbol{e^{1/2022}}$

   Solution: Define $f(x)$ to be the expected value of the additonal number of items Sumith buys, given that he has already spent x. Our goal is to find $f(0)$. The key idea is to frame this as a states problem: note $f(x) = 0$ for $x > 1$, and for any x with $0 \le x \le 1$, we have

$$f(x) = 1 + \frac{1}{2022}\int_x^{x+2022} f(x)\,\mathrm{d}x = 1 + \frac{1}{2022}\int_x^1 f(x)\,\mathrm{d}x\,.$$

   Taking the derivative of both sides, we obtain a differential equation

$$f'(x) = -\frac{f(x)}{2022}$$

   which has solution $f(x) = Ce^{-x/2022}$ where C is some real-number constant. Because $f(1) = 1$, we must have

$$f(x) = e^{(1-x)/2022}.$$

   Therefore, our answer is $f(0) = \boxed{e^{1/2022}}$.

9. Compute

$$\int_0^{\frac{\pi}{2}} \cot(x)\ln(\cos(x))\,\mathrm{d}x,$$

where ln denotes the natural logarithm.

Answer: $-\frac{\pi^2}{24}$

Solution: Notice that we can manipulate

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cot(x)\ln(\cos(x))\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{\cos(x)\ln(\cos(x))}{\sin(x)}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos(x)\ln\left(\cos^2(x)\right)}{2\sin(x)}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos(x)\ln\left(1-\sin^2(x)\right)}{2\sin(x)}\,\mathrm{d}x.\end{aligned}$$

Consider the u-substitution $u = \sin(x)$. Then $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, so

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)\ln\left(1-\sin^2(x)\right)}{2\sin(x)}\,\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{\ln\left(1-u^2\right)}{u}\,\mathrm{d}u\,.$$

Using the Taylor expansion $\ln(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}$, we get

$$\begin{aligned}\frac{1}{2}\int_0^1 \frac{\ln\left(1-u^2\right)}{u}\,\mathrm{d}u &= -\frac{1}{2}\int_0^1 \frac{1}{u}\left(\sum_{k=1}^{\infty}\frac{u^{2k}}{k}\right)\mathrm{d}u \\ &= -\frac{1}{2}\int_0^1 \left(\sum_{k=1}^{\infty}\frac{u^{2k-1}}{k}\right)\mathrm{d}u.\end{aligned}$$

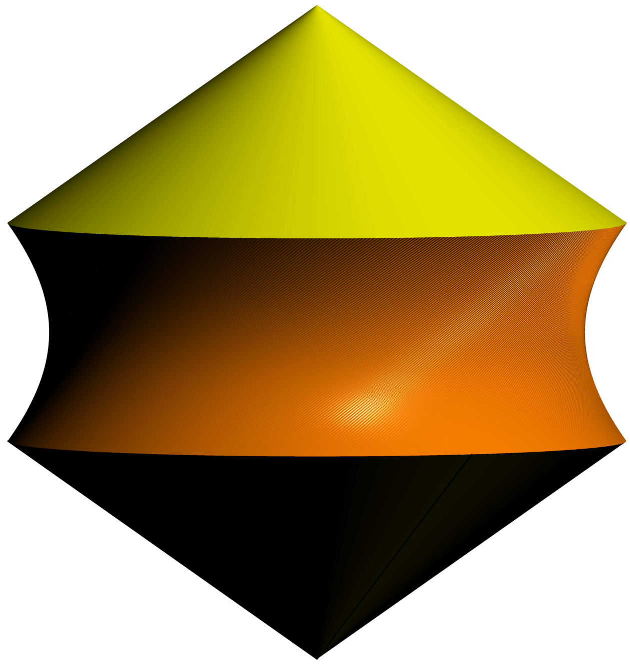
We may flip the integral and summation sign (this is intuitively clear but can be made rigorous due to a result from real analysis), which gives

$$\begin{aligned}-\frac{1}{2}\int_0^1 \left(\sum_{k=1}^{\infty}\frac{u^{2k-1}}{k}\right)\mathrm{d}u &= -\frac{1}{2}\sum_{k=1}^{\infty}\left(\int_0^1 \frac{u^{2k-1}}{k}\,\mathrm{d}u\right) \\ &= -\frac{1}{4}\sum_{k=1}^{\infty}\left[\frac{u^{2k}}{k^2}\right]_0^1 \\ &= -\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} \\ &= \boxed{-\frac{\pi^2}{24}}.\end{aligned}$$

10. A unit cube is rotated around an axis containing its longest diagonal. Compute the volume swept out by the rotating cube.

Answer: $\frac{\pi}{\sqrt{3}}$

Solution: Let the corners of the cube be represented by the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $\ldots$, $(1,1,1)$, and say that we are rotating the cube about line L, which contains $(0,0,0)$ and $(1,1,1)$. Here is the image, where L has been aligned vertically.



This solid of revolution will consist of three parts: a cone (yellow), a middle part (orange), and a cone (shaded). By symmetry, the volume of the two cones are the same, so we only need to find the volumes of the middle part and one cone. We define $f(x)$ to be the radius of the solid at a distance x above $(0, 0, 0)$ along the axis. Note that the corresponding point to rotate around is $\frac{1}{\sqrt{3}}(x, x, x)$.

We now split the volume computations into two parts.

- We compute the volume of the middle part. By symmetry, we can see that the surface of the middle part will be traced out by rotating the segment containing $(1, 0, 0)$ and $(1, 1, 0)$ about the axis. First of all, we parametrize this line segment as $(1, t, 0)$ for $0 \le t \le 1$.

  Now, we fix t and define x_t such that $\frac{1}{\sqrt{3}}(x_t, x_t, x_t)$ is the closest point on L to $(1, t, 0)$. In particular, note that

  $$f(x_t)^2 = \left\| (1, t, 0) - \frac{1}{\sqrt{3}}(x_t, x_t, x_t) \right\|,$$

  which lets us find $f(x)$ for the middle part; here, $\|(a, b, c)\| = a^2 + b^2 + c^2$. For the fixed t, we will find $f(x_t)^2$ and the corresponding x_t by noting

  $$f(x_t)^2 = \min_{x \in [0, \sqrt{3}]} \left\| (1, t, 0) - \frac{1}{\sqrt{3}}(x, x, x) \right\|.$$

  Indeed, to simplify the expression, set $y = \frac{x}{\sqrt{3}}$, which gives

  $$\begin{aligned} f(x_t)^2 &= \min_{y \in [0,1]} \left((1 - y)^2 + (t - y)^2 + y^2 \right) \\ &= \min_{y \in [0,1]} \left(3y^2 - 2y(t + 1) + t^2 + 1 \right). \end{aligned}$$

  We find the minimum by taking the derivative and setting to 0, which gives $y_t = \frac{1+t}{3}$. Plugging this, we solve

  $$f(x_t)^2 = \frac{2}{3}(t^2 - t + 1).$$

Lastly, we extract t: because $y_t = \frac{1+t}{3}$, we see $x_t = \frac{1+t}{\sqrt{3}}$. Thus, $t = x_t\sqrt{3} - 1$, so in fact

$$f(x_t)^2 = 2x_t^2 - 2x_t\sqrt{3} + 2,$$

where $0 \le t \le 1$.

The above equation defines $f(x)$ for the middle part, which we can now see is over the interval $[x_0, x_1] = [1/\sqrt{3}, 2/\sqrt{3}]$. We now may compute the volume of the middle part as

$$\begin{aligned} V_{\text{middle}} &= \int_{x_0}^{x_1} \pi f(x)^2 \, \mathrm{d}x \\ &= \pi \int_{1/\sqrt{3}}^{2/\sqrt{3}} \left(2x^2 - 2x\sqrt{3} + 2\right) \mathrm{d}x \\ &= \pi \left(\frac{2}{3}\left[x^3\right]_{1/\sqrt{3}}^{2/\sqrt{3}} - \sqrt{3}\left[x^2\right]_{1/\sqrt{3}}^{2/\sqrt{3}} + 2[x]_{1/\sqrt{3}}^{2/\sqrt{3}}\right) \\ &= \pi \left(\frac{2}{3} \cdot \frac{7}{3\sqrt{3}} - \sqrt{3} \cdot \frac{3}{3} + 2 \cdot \frac{1}{\sqrt{3}}\right) \\ &= \frac{5\pi}{9\sqrt{3}}. \end{aligned}$$

- We compute the volume of the cones. Each of the cones has height x_0 and radius $f(x_0)$, where $x_0 = 1/\sqrt{3}$ with $f(x_0)^2 = 2/3$ was computed above. Thus, the volume of the cones is

$$\begin{aligned} V_{\text{cones}} &= 2 \cdot \frac{\pi}{3} \cdot f(x_0)^2 \cdot x_0 \\ &= \frac{2\pi}{3} \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} \\ &= \frac{4\pi}{9\sqrt{3}}. \end{aligned}$$

Summing, the volume of the entire solid is

$$V = V_{\text{cones}} + V_{\text{middle}} = \frac{5\pi}{9\sqrt{3}} + \frac{4\pi}{9\sqrt{3}} = \boxed{\frac{\pi}{\sqrt{3}}}.$$