1. Compute
$$\lim_{x\to 0}\left(\frac{\tan^{-1}(x)}{\tan^{-1}(x)+x}\right)^3.$$

Answer: $\frac{1}{8}$

Solution: So this function is nasty, let's kind of cheat a little bit.
$$\lim_{x\to 0}\left(\frac{\tan^{-1}(x)+x}{\tan^{-1}(x)}\right)=\lim_{x\to 0}\left(1+\frac{x}{\tan^{-1}(x)}\right)=2$$
$$\lim_{x\to 0}\left(\frac{\tan^{-1}(x)+x}{\tan^{-1}(x)}\right)^3=\lim_{x\to 0}\left(\frac{1}{2}\right)^3=\boxed{\frac{1}{8}}$$

2. Compute
$$\int_0^1 \frac{x^2}{1+\sqrt{1-x^2}}\,\mathrm{d}x\,.$$

Answer: $\frac{4-\pi}{4}$

Solution: We make the trig substitution $x=\sin\theta$.

We obtain
$$\begin{aligned}\int_0^{\pi/2}\frac{\sin^2\theta\cos\theta}{1+\cos\theta}d\theta &= \int_0^{\pi/2}\left(\frac{\sin\theta}{1+\cos\theta}\right)\sin\theta\cos\theta d\theta\\ &= \int_0^{\pi/2}\tan(\theta/2)\sin\theta\cos\theta d\theta\\ &= \int_0^{\pi/2}2\tan(\theta/2)\sin(\theta/2)\cos(\theta/2)\cos\theta d\theta\\ &= 2\int_0^{\pi/2}\sin^2(\theta/2)\cos\theta d\theta\\ &= 2\int_0^{\pi/2}\sin^2(\theta/2)(1-2\sin^2(\theta/2))d\theta\\ &= 2\int_0^{\pi/2}\sin^2(\theta/2)d\theta-4\int_0^{\pi/2}\sin^4(\theta/2)d\theta\\ &= 2\int_0^{\pi/2}\sin^2(\theta/2)d\theta-4\int_0^{\pi/2}\sin^2(\theta/2)(1-\cos^2(\theta/2))d\theta\\ &= \int_0^{\pi/2}4\sin^2(\theta/2)\cos^2(\theta/2)-2\int_0^{\pi/2}\sin^2(\theta/2)d\theta\\ &= \int_0^{\pi/2}\sin^2(\theta)d\theta-2\int_0^{\pi/2}\sin^2(\theta/2)d\theta\end{aligned}$$

Then, we evaluate each integral as follows:
$$\begin{aligned}\int_0^{\pi/2}\sin^2(\theta/2)d\theta &= \int_0^{\pi/2}\frac{1-\cos\theta}{2}d\theta\\ &= \frac{\pi}{4}-\frac{1}{2}\int_0^{\pi/2}\cos\theta d\theta\\ &= \frac{\pi}{4}-\frac{1}{2},\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/2} \sin^2(\theta) &= \int_0^{\pi/2} \frac{1-\cos(2\theta)}{2} d\theta \\ &= \frac{\pi}{4} - \frac{1}{2}\int_0^{\pi/2} \cos(2\theta) d\theta \\ &= \frac{\pi}{4}.\end{aligned}$$

Thus, the integral is

$$\int_0^1 \frac{x^2}{1+\sqrt{1-x^2}} dx = \frac{\pi}{4} - 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \boxed{\frac{4-\pi}{4}}.$$

3. Compute

$$\sum_{n=0}^{\infty} \frac{n^3}{n!}.$$

Answer: $5e$

Solution: The series is reminiscent of the power series definition for e^x. Our strategy will be to repeatedly take the derivative of the power series of e^x and multiply by x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Take the derivative and multiply by x:

$$e^x = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!}$$

$$xe^x = \sum_{n=1}^{\infty} \frac{nx^n}{n!} = \sum_{n=0}^{\infty} \frac{nx^n}{n!}.$$

Repeat:

$$e^x + xe^x = \sum_{n=1}^{\infty} \frac{n^2x^{n-1}}{n!}$$

$$xe^x + x^2e^x = \sum_{n=0}^{\infty} \frac{n^2x^n}{n!}.$$

Repeat one last time:

$$e^x + 3xe^x + x^2e^x = \sum_{n=1}^{\infty} \frac{n^3x^{n-1}}{n!}$$

$$xe^x + 3x^2e^x + x^3e^x = \sum_{n=0}^{\infty} \frac{n^3x^n}{n!}.$$

We plug in $x = 1$ to get:

$$\sum_{n=0}^{\infty} \frac{n^3x^n}{n!} = \boxed{5e}.$$